# 2017

## STATISTICS

( Major )

Paper : 3.1

## ( Mathematical Methods—II )

*Full Marks* : 60

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

**1.** Answer the following questions : $1 \times 7 = 7$

(a) Suppose $A$ is a $(4 \times 5)$ matrix and $\rho(A) = 3$ (i.e., the rank of the matrix $A$). Let $A_{44}$ be a fourth order minor of $A$. Find $\det(A_{44})$.

(b) Let $A$ and $B$ be two matrices such that

$$A = \begin{pmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 9 & -1 & 6 & -3 \\ 36 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

It is given that $\rho(A) = 4$. Find $\rho(B)$.

*(c)* For the following system of homogeneous equations, i.e.,

$$(A)_{n\times n}(X)_{n\times 1}=(0)_{n\times 1}$$

it is given that $\rho(A)=n$. Find the number of linearly independent solutions that this system of equations will possess.

*(d)* Suppose $A$ is a $(6\times 6)$ idempotent matrix given $\rho(I-A)=4$ ($I$ is a $6\times 6$ identity matrix). Find $\rho(A)$.

*(e)* Express the vector $\begin{pmatrix}5\\7\\8\end{pmatrix}$ as a linear combination of the vectors

$$\begin{pmatrix}1\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\0\\1\end{pmatrix}$$

*(f)* Given that $A$ is a $(5\times 5)$ matrix and is non-singular. Find $\rho(A)$.

*(g)* Consider a linear transformation which transforms the quadratic form $Q=X'AX$ into $Y'IY$, where $X=(X_1, X_2, ..., X_n)^T$, $A$ is an $n\times n$ matrix, $Y=(Y_1, Y_2, ..., Y_n)^T$ and $I=\begin{pmatrix}I_r & 0\\ 0 & 0\end{pmatrix}_{n\times n}$, $r<n$ (i.e., $I_r$ is an identity matrix of order $r$). Identify whether $Q$ is a positive definite or positive semi-definite quadratic form.

**2.** Answer the following : 2×4=8

*(a)* Given the matrix

$$A = \begin{pmatrix} 2 & 9 & 6 \\ 8 & 7 & 5 \\ 11 & 9 & 2 \end{pmatrix}$$

Suppose a row transformation has been applied to it to obtain $B = \begin{pmatrix} 11 & 9 & 2 \\ 8 & 7 & 5 \\ 2 & 9 & 6 \end{pmatrix}$

($A$ and $B$ are similar). Identify an appropriate elementary matrix such that multiplication of $A$ by it would give $B$.

*(b)* Given for a $(3 \times 3)$ matrix $|\text{adj}A| = 20$. Find $|A|$.

*(c)* Show that if $A$ and $B$ are two unitary matrices, then their product $AB$ will also be unitary.

*(d)* Consider the system of non-homogeneous linear equations

$$(A)_{m \times n} (X)_{n \times 1} = (B)_{m \times 1}$$

It is given that the system is consistent and $\rho(A) = 5 \le \min(m, n)$. Find the rank of the augmented matrix $(A : B)$, i.e., $\rho(A : B)$.

**3.** Answer any *three* of the following : 5×3=15

(a) State the three elementary operations on matrices. Show with an example, how an elementary row transformation can be implemented by pre-multiplication of the matrix with an appropriate elementary matrix.

(b) Show that for any two matrices $A$ and $B$

$$\rho(AB) \leq \min(\rho(A),\ \rho(B))$$

(c) Compute the inverse of the following matrix by using elementary operations :

$$\begin{pmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

(d) Prove that the necessary and sufficient condition for a system of $n$ homogeneous equations in $n$ variables, i.e.,

$$(A)_{n\times n}(X)_{n\times 1} = (0)_{n\times 1}$$

to possess a non-trivial solution is that $|A| = 0$ (i.e., $\det(A) = 0$).

(e) Suppose $x_1, x_2, \ldots, x_n$ is a random sample of size $n$ from a given population. Express the sample variance for this sample as a quadratic form. Identify a quadratic form in the multivariate normal distribution.

**4.** Answer any *three* of the following : 10×3=30

*(a)* Prove that the rank of a matrix $A$ remains unchanged by premultiplication (or post-multiplication) with a non-singular matrix, i.e., $\rho(PA) = \rho(A) = \rho(AQ)$, where $P$ and $Q$ are non-singular matrices of appropriate order. 10

*(b)* What is meant by an orthogonal matrix? Show that if $A$ and $B$ are two $n$-rowed orthogonal matrices, then $AB$ is also an orthogonal matrix. How is an orthogonal matrix different from an unitary matrix? Show that the following matrix

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

is unitary $(i^2 = -1)$. 2+4+2+2=10

*(c)* *(i)* Suppose $A$ is an $(m \times n)$ matrix and $B$ is an $(n \times p)$ matrix and are partitioned as

$$A = \begin{pmatrix} R_1 \\ R_2 \\ R_m \end{pmatrix}_{m \times 1} \text{ and } B = (C_1, C_2, ..., C_p)_{1 \times p}$$

where $R_i$ is the $i$th row of $A$ and $C_j$ is the $j$th column of $B$ $(i = 1, 2, ..., m$ and $j = 1, 2, ..., p)$. Obtain $AB$ in terms of $R_i$'s and $C_j$'s.

*(ii)* Given the matrix

$$M = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 5 & 2 \\ 1 & 9 & 7 \\ 3 & -2 & 8 \end{pmatrix}$$

Obtain all possible minors of $M$.

6+4=10

*(d)* *(i)* Investigate for what values of $\lambda$ and $\mu$, the simultaneous equations

$$x + y + z = 6,\ x + 2y + 3z = 10 \text{ and } x + 2y + \lambda z = \mu$$

have an infinite number of solutions.

*(ii)* Show that the following system of equations

$$x + y + z = -3,\ 3x + y - 2z = -2 \text{ and } 2x + 4y + 7z = 7$$

are not consistent.

6+4=10

*(e)* *(i)* Find the inverse of the matrix $A$ and hence solve the equation $AX^T Y$, where

$$A = \begin{pmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix},\ X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \text{ and } Y = \begin{pmatrix} -1 \\ 1 \\ -5 \end{pmatrix}$$

*(ii)* Show that

$$B = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

is the inverse of $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

6+4=10

*(f)* Consider the following quadratic form :

$$Q = 6X_1^2 + 3X_2^2 + 14X_3^2 + 4X_2X_3 + 18X_3X_1 + 4X_1X_2$$

Find a transformation $Y = AX$, where $Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$, $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ and $A$ is a $(3 \times 3)$ matrix which reduces it to the form $6Y_1^2 - 29Y_2^2 + Y_3^2$. Also identify the index and signature of the quadratic form. 10

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